ON AN EXOTIC LAGRANGIAN TORUS IN $\mathbb{C}P^2$

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Abstract: We find a non-displaceable Lagrangian torus fiber in a semi-toric system which is superheavy with respect to certain symplectic quasi-state. In particular, this proves Lagrangian $\mathbb{R}P^2$ is not a stem in $\mathbb{C}P^2$, answering a question of Entov and Polterovich.

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1. Introduction

The primary goal of this paper is to understand a toric degeneration model of $\mathbb{C}P^2$. This model comes from the toric degeneration of $S^2 \times S^2$, which is the same as the symplectic cut on a level set of T^*S^2, see Section 3. The degenerated torus action still gives a moment polytope , which is $P_{S^2\times S^2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + 2x_2 \leq 2\} \subset \mathbb{R}^2$ (Figure 1). If one considers a similar picture on $T^*\mathbb{R}P^2$, the symplectic cut on a level set leads to $\mathbb{C}P^2$. The corresponding moment polytope is as in Figure 2, and can be described as $P_{\mathbb{C}P^2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + 4x_2 \leq 4\}$.

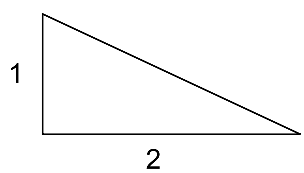

Figure 1. $S^2 \times S^2$

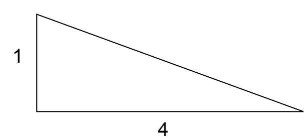

Figure 2. $\mathbb{C}P^2$

In [21], the authors there considered the Floer theory of smooth fibers in the toric degeneration model of $S^2 \times S^2$, proving by bulk deformation that there are uncountably many non-displaceable fibers in Figure 1. In view of Albers-Frauenfelder's result [2], we may interpret this result as that, only non-displaceable torus fibers below the "monotone level" in the semi-toric system survives the symplectic cut along a level set of T^*S^2. This implies that the anti-diagonal of $S^2 \times S^2$ is not a stem (see 2.4.1 for the definition of a stem), answering a question raised by Entov and Polterovich in [14]. This was also proved independently by several other authors [13, 8]. In [13] it was mentioned that Wehrheim also has an unpublished note on this problem.

From Fukaya-Oh-Ohta-Ono's calculation on $S^2 \times S^2$, we expect the above similar picture of $\mathbb{C}P^2$ contains uncountably many non-displaceable fibers. This corresponds to an easy adaption of Albers-Frauenfelder's result to $T^*\mathbb{R}P^2$, by considering the $\mathbb{Z}_2$-involution induced by antipodal map on S^2.

In this paper we find one smooth non-displaceable monotone torus fiber in the moment polytope described above, and prove that it is superheavy with respect to some partial symplectic quasi-state. In particular, we proved:

Theorem 1.1. *There is a smooth monotone Lagrangian torus fiber in Figure 2, which is superheavy with respect to certain partial symplectic quasi-state. In particular, it is stably non-displaceable.*

The limitation to such a monotone fiber is due to the difficulty of using $\mathbb{R}P^2$ as a bulk to deform our Floer cohomology, since it is only a non-trivial $\mathbb{Z}_2$-class. The author does not know whether this is only technical. Nonetheless, this result suffices to show:

Corollary 1.2. *$\mathbb{R}P^2 \subset \mathbb{C}P^2$ is not a stem.*

This answers the question of Entov-Polterovich ([15], 9.2) regarding the case of $\mathbb{C}P^2$.

Remark 1.3. It is likely that our exotic monotone Lagrangian is the one described by Chekanov-Schlenk in $\mathbb{C}P^2$, or at least there is certain relations between them. It would be nice if the relation between the two can be cleared up.

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2. Preliminaries

The current section summarizes part of the Lagrangian Floer theory developed by Fukaya-Oh-Ohta-Ono [18, 19, 20], as well as the theory of symplectic quasi-states developed by Entov and Polterovich in a series of their works [16, 14, 15]. The aim of this section is to recall basic notions and main framework results in these theories for our applications, as well as for the convenience of readers. Therefore, our scope is rather restricted and will not provide a thorough account to the whole theory. For details and proofs one is refered to the above-mentioned works. Much of our discussions on Lagrangian Floer theory follow the lines of the survey part in Section 2 of [21].

2.1. **Moduli spaces of border holomorphic curves.** Let (M,ω) be a smooth symplectic manifold and $L \subset M$ a relatively spin Lagrangian. This means the second Stiefel-Whitney class $w_2(L)$ is in the image of the restriction map $H^2(M,\mathbb{Z}_2) \to H^2(L,\mathbb{Z}_2)$. Let $J \in \mathcal{J}_\omega$, the space of compatible almost complex structures, and $\beta \in H_2(M,L;\mathbb{Z})$. We denote by $\mathcal{M}^{\text{main}}_{k+1,l}(\beta; M.L; J)$ as the space of J-holomorphic bordered stable maps in class β with $k+1$ boundary marked points and l interior marked points. Here, we require the boundary marked points to be ordered counter-clockwisely. When no confusions is likely to occur, we will suppress M, L or J.

One of the fundamental results in [18] shows that, one has a Kuranishi structure on $\mathcal{M}^{\text{main}}_{k+1,l}(\beta, L)$, so that the evaluation maps at the i^{th} boundary marked point (j^{th} interior marked point, respectively)

$$ev_i : \mathcal{M}^{\text{main}}_{k+1,l}(\beta, L) \to L,$$

and

$$ev_j^+ : \mathcal{M}^{\text{main}}_{k+1,l}(\beta, L) \to M$$

are weakly submersive (see [18] for the definition of weakly submersive Kuranishi maps). For given smooth singular simplices $(f_i : P_i \to L)$ of L and $(g_j : Q_j \to M)$ of M, one can also define the fiber product in the sense of Kuranishi structure:

$$\mathcal{M}^{\text{main}}_{k+1,l}(\beta; L; \overrightarrow{Q}, \overrightarrow{P}) := \mathcal{M}^{\text{main}}_{k+1,l}(\beta; L)_{(ev_1^+,\dots,ev_l^+,ev_1,\dots,ev_k)\times(g_1,\dots,g_l,f_1,\dots,f_k)} (\Pi_{j=1}^l Q_j \times \Pi_{i=1}^k P_i).$$

The virtual fundamental chain associated to this moduli space,

$$ev_0 : \mathcal{M}^{\text{main}}_{k+1,l}(\beta; L; \overrightarrow{Q}, \overrightarrow{P}) \to L$$

as a singular chain, is defined in [18] via techniques of virtual perturbations.

2.2. **A_∞-structures and potential functions.** In [18], the authors there discovered that, unlike the case of (weakly) exact Lagrangian submanifolds or Hamiltonian Floer theory, a homology theory associated to a general Lagrangian submanifold is usually not well-defined if one attempts to generalize the theory in the obvious way. The appropriate algebraic model, instead, is the filtered A_∞-algebra.

Consider the universal Novikov rings:

$$\Lambda = \{\sum a_i T^{\lambda_i} | a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty\},$$
$$\Lambda_0 = \{\sum a_i T^{\lambda_i} | a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i \leq \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = \infty\}.$$

Here T is a formal variable. Consider a valuation which assigns $\sigma_T(\sum a_i T^{\lambda_i}) = \lambda_1$ if not all $a_i = 0$, and let $\sigma_T(0) = +\infty$. This induces a $\mathbb{R}$-filtration on Λ and Λ_0 thus a non-Archimedian topology. Note that $\Lambda_0 \subset \Lambda$, and Λ_0 has a maximal ideal Λ_+ consisting of elements with $\lambda_i > 0$ for all i.

We now define the A_∞-structure for a Lagrangian $L \subset M$. Let $P, P_i \subset C^*(L, \mathbb{C})$ be singular simplices and $\beta \in H_2(X, L; \mathbb{Z})$. Here $C^*(L;\mathbb{C}) = \oplus C^k(L;\mathbb{C})$, where $C^k(L;\mathbb{C})$ is generated by the codimension k singular simplices of L. We may define the following singular chains in L.

Definition 2.1.

$$\mathfrak{m}_{0,\beta}(1) = \begin{cases} (\mathcal{M}_1(\beta; L), ev_0), & \beta \neq 0 \\ 0, & \beta = 0 \end{cases}$$

$$\mathfrak{m}_{1,\beta}(P) = \begin{cases} (\mathcal{M}_1(\beta; P; L), ev_0), & \beta \neq 0 \\ (-1)^n \partial P, & \beta = 0 \end{cases}$$

$$\mathfrak{m}_{k,\beta}(P_1, \dots, P_k) = (\mathcal{M}^{\mathrm{main}}_{k+1,l}(\beta; P_1, \dots, P_k), ev_0), \quad k \geq 2.$$

Theorem 3.5.11 in [18] showed that there is a subcomplex $C' \subset C_*(L, \mathbb{C})$ so that $\mathfrak{m}_{0,\beta}(1), \mathfrak{m}_{k,\beta}(P_1, \dots, P_k) \in C'$ whenever $P_1, \dots, P_k \in C'$. For $PT^\lambda \in C$, define $\deg(PT^\lambda) = n - dimP$. Let $B_k(C[1])$ be the completion with respect to σ_T of the k-fold tensor product $\bigotimes_{i=1}^k C[1]$. We may now define the multiplication $\mathfrak{m}_k : B_k(C[1]) \to C[1]$ by:

$$\mathfrak{m}_k = \sum_{\beta \in H_2(M,L;\mathbb{Z})} \mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)/2\pi}.$$

One may extend $\mathfrak{m}_k$ to a graded coderivation $\hat{\mathfrak{m}}_k$ on $B(C[1]) = \bigoplus_{k=1}^\infty B_k(C[1])$ and define $\hat{d} = \sum_{k=1}^\infty \hat{\mathfrak{m}}_k : B(C[1]) \to B(C[1])$ (see their definitions in [18]). The *filtered A_∞-condition* requries that $\hat{d} \circ \hat{d} = 0$.

For $b \in C[1]^0$, $b \equiv 0 \bmod \Lambda_+$, the A_∞-multiplications can be deformed as:

$$\mathfrak{m}^b_k(x_1, \dots, x_k) = \sum_{l=0}^\infty \mathfrak{m}_{k+l}(b, \dots, b, x_1, b, \dots, b, x_2, b, \dots, b, x_k, b, \dots, b)$$

where the insertion of b's takes arbitrary length starting from 0, and l is the total number of b-insertions. We may then define the deformed coderivation $\hat{d}^b$ similarly.

Theorem 2.2 ([18])**.** *For any closed relatively spin Lagrangian submanifolds $L \subset M$, there exists a countably generated subcomplex C of smooth singular cochain complex of L, whose cohomology is isomorphic to $H^*(L;\mathbb{C})$, along with a system of multi-valued sections of Kuranishi structure of $\mathcal{M}^{main}_{k+1}(\beta; P_1, \dots, P_k)$, such that $\mathfrak{m}_k$ are defined as above and the associated $\hat{d}$ satisfies $\hat{d}^2 = 0$. Therefore, $(C, \mathfrak{m}_k)$ forms a filtered A_∞-algebra. Similar statements holds also for $(C, \mathfrak{m}^b_k)$ for $b \in \Lambda_0^+$.*

To make our filtered A_∞-algebra "smaller", one further has:

Theorem 2.3 ([18])**.** *In the same situation above, we can construct a filtered A_∞-algebra structure $\mathfrak{m}^{can} = \{\mathfrak{m}_k^{can}\}$ on $H(L;\Lambda_0)$ so that it is homotopy equivalent to $(C, \mathfrak{m})$ that we constructed above.*

To extend our deformations to the whole $H^1(L;\Lambda_0)$, we use an idea due to Cho [9]. Let $b_0 \in H^1(L;\mathbb{C})$, and let $\rho_{b_0} : H_1(L;\mathbb{C}) \to \mathbb{C}^*$ be a representative defined by

$$\rho_{b_0}(\gamma) = exp \int_\gamma b_0,$$

for $[\gamma] \in H_1(L;\mathbb{C})$. We now define

$$\mathfrak{m}_k^{b_0} = \sum_\beta \mathfrak{m}_{k,\beta}^{b_0} \otimes T^{\omega(\beta)/2\pi} = \sum_{k,\beta} \rho_{b_0}(\partial\beta)\mathfrak{m}_{k,\beta} \otimes T^{\omega(\beta)/2\pi}.$$

For $b = b_0 + b_+ \in H^1(L;\mathbb{C}) \oplus H^1(L;\Lambda_+) = H^1(L;\Lambda_0)$, put

$$\mathfrak{m}_{k,\beta}^b(x_1, \dots, x_k) = \sum_l \mathfrak{m}_{k+l,\beta}^{b_0}(b_+, \dots, b_+, x_1, b_+, \dots, b_+, x_k, b_+, \dots b_+),$$

and $\mathfrak{m}_k^b = \sum \mathfrak{m}_{k,\beta}^b \otimes T^{\omega(\beta)/2\pi}$. Then $(H(L,\Lambda_0), \mathfrak{m}_k^b)$ also has the structure of filtered A_∞-algebra.

Due to the presence of the $\mathfrak{m}_0^b$-term, which is sometimes called the *central charge*, one does not have $(\mathfrak{m}_1^b)^2 = 0$ in general. Therefore, to have a well-defined Floer cohomology theory, we introduce:

Definition 2.4. $b = b_0 + b_+ \in H^1(L;\mathbb{C}) \oplus H^1(L;\Lambda_+) = H^1(L;\Lambda_0)$ is called a *weak bounding cochain*, if it satisfies the weak Maurer-Cartan equation:

$$\sum_k^{b_0} \mathfrak{m}_k^{b_0}(b_+, \dots, b_+) = cPD([L]),$$

for some $c \in \Lambda_+$. Denote by $\widehat{\mathcal{M}}_{\text{weak}}(L, \mathfrak{m})$ the set of weak bounding cochains, and take c as the value of a function $PD^L : \widehat{\mathcal{M}}_{\text{weak}}(L, \mathfrak{m}) \to \Lambda_+$. This function is called the *potential function of L*.

We now can define for $b^0, b^1 \in \widehat{\mathcal{M}}_{\text{weak}}(L, \mathfrak{m})$ an operator

$$\delta_{b^1,b^0} : H(L;\Lambda_0) \to H(L;\Lambda_0)$$

by $\delta_{b^1,b^0}(x) = \sum_{k_0,k_1} \mathfrak{m}_{k_0+k_1+1}(b^1, \dots, b^1, x, b^0, \dots, b^0)$, which verifies $(\delta_{b^0,b^1})^2(x) = (-PD^L(b^1) + PD^L(b^0)) \cdot x$. Therefore,

Definition 2.5. In the situation above, if $PD^L(b^1) = PD^L(b^0)$, we define

$$HF((L, b^1), (L, b^0)) = \mathrm{Ker}(\delta_{b^1,b^0})/\mathrm{Im}(\delta_{b^1,b^0}),$$

which we call the Floer cohomology of L deformed by (b^1, b^0).

2.3. **Toric case of Lagrangian Floer theory.** When M is a smooth toric manifold, let $\dim_{\mathbb{C}} M = n$ and $\Phi : M \to P$ be its moment map. The main concern of ourstudy is the Lagrangian toric fibers, denoted $L(u) = \Phi^{-1}(u)$, $u \in P$. The first proposition we have is:

Proposition 2.6 ([19])**.**

$$H^1(L(u);\Lambda_0)/H^1(L(u);2\pi\sqrt{-1}\mathbb{Z}) \hookrightarrow \widehat{\mathcal{M}}_{weak}(L(u), \mathfrak{m}).$$

We would like to simplify our expression of potential function by restricting our attention to this subset of weak bounding cochains. Choose an integral basis e_i^*, $i = 1, \dots, n$ of $H_1(L(u);\mathbb{Z})$ with dual basis e_i of $H^1(L(u);\mathbb{Z})$. For $b = \sum_{i=1}^n x_i e_i \in H^1(L(u);\Lambda_0)$, $x_0 \in \Lambda_0$ are regarded as coordinates. The potential function is now written as:

$$PD^u(x_1, \dots, x_n) : (\Lambda_0/2\pi\sqrt{-1}\mathbb{Z})^n \cong (\Lambda\backslash\Lambda_+)^n \longrightarrow \Lambda_+.$$

Here the isomorphism on the left hand side is taken as the exponential function. This change of coordinates will be convenient for us. Regard $y_i = e^{x_i}$, then PD^u becomes a function of y_i. Without the assumption that (M, ω) being toric, we have:

Theorem 2.7 (Theorem 2.3, [21])**.** *Let $L \subset (M, \omega)$ be a Lagrangian torus and assume $H^1(L(u); \Lambda_0)/H^1(L(u); 2\pi\sqrt{-1}\mathbb{Z}) \hookrightarrow \widehat{\mathcal{M}}_{weak}(L(u), \mathfrak{m})$. If $b \in H^1(L; \Lambda_0)$ is a critical point of PD^L, then*

$$HF((L, b), (L, b); \Lambda_0) \cong H(L; \Lambda_0).$$

In particular, L is non-displaceable.

2.4. **Symplectic quasi-states and Lagrangian Floer theory.**

2.4.1. *Symplectic quasi-states, superheaviness and stems.* In this section we briefly review the theory of (partial) symplectic quasi-states developed by Entov and Polterovich. *Partial symplectic quasi-states* are functionals $\zeta : C^\infty(M) \to \mathbb{R}$ satisfying the following axioms for $H, K \in C^\infty(M)$ and $\alpha \in \mathbb{R}_{\geq 0}$:

(i) (Additivity with constants and normalization) $\zeta(H + \alpha) = \zeta(H) + \alpha$. In particular $\zeta(1) = 1$;
(ii) (Monotonicity) If $H \leq K$, then $\zeta(H) \leq \zeta(K)$;
(iii) (Triangle inequality) If $\{H, K\} = 0$, then $\zeta(H + K) \leq \zeta(H) + \zeta(K)$;
(iv) (Partial additivity) If $\{H, K\} = 0$, and $\mathrm{supp}(H)$ is displaceable, then $\zeta(H + K) = \zeta(K)$. In particular $\zeta(H) = 0$;
(v) (Hamiltonian invariance) $\zeta(H) = \zeta(H \circ f)$ for $f \in Symp_0(M)$

Given a partial symplectic quasi-state ζ and a subset $S \subset M$, S is called ζ-*heavy* if:

$$\zeta(F) \geq \inf_{x \in S} F(x), \quad \forall F \in C^\infty(M);$$

and ζ-*superheavy* if

$$\zeta(F) \leq \sup_{x \in S} F(x), \quad \forall F \in C^\infty(M).$$

One of the key properties of these subsets proved in [14] is that, a ζ-superheavy subset is always ζ-heavy, and must intersect a ζ-heavy subset. Combining with the Hamiltonian invariance property, one deduces the non-displaceability of a superheavy subset. Let $V \subset C^\infty(M)$ be a finite dimensional linear subspace spanned by pairwisely Poisson-commuting functions. Let $\Psi : M \to V^*$ be the moment map defined by $\langle \Psi(x), F \rangle = F(x)$ for $F \in V$. A non-empty fiber of this moment map is called a *stem*, if the rest of the fibers are all displaceable. It was essentially proved in Theorem 1.6 of [14] the following:

Theorem 2.8 ([14])**.** *A stem is a superheavy subset with respect to arbitrary partial symplectic quasi-states.*

We will use the canonical construction of partial symplectic quasi-state in the rest of the paper. Using spectral invariants, it was proved in [14] that there is a partial symplectic quasi-state associated to an idempotent $e \in QH^*(M; \Lambda)$. We will denote such a symplectic quasi-state as ζ_e. We will explain certain properties of ζ_e useful for us in subsequent sections.

2.4.2. *Relations to Lagrangian Floer theory.* In this section, we will explain a theorem due to Fukaya-Oh-Ohta-Ono which relates superheaviness and Lagrangian Floer cohomology. Some of the expositions will only be a sketch in considerations to be concise, and the readers are referred to [18] for full details.

For V a linear space, denote $E_l(V)$ to be the invariant elements of $B_l(V)$ under the obvious symmetric group action. Similar to the definition of $\mathfrak{m}_{\beta;k,l}$, we choose an appropriate subcomplex $C(M)$ and $C(L)$ of the singular chain complex of M and L, respectively. One may then define an operator $q_{\beta;l,k} : E_l C(M) \otimes B_k C(L) \to C(L)$ using results stated in Section 2.1. Namely, we let

$$q_{\beta;l,k}(\overrightarrow{Q}, \overrightarrow{P}) = \frac{1}{l!}(\mathcal{M}^{main}_{k+1}(L; \beta; \overrightarrow{Q}, \overrightarrow{P}), ev_0).$$

For $\overrightarrow{Q} \in E_l(C(M))$, $\overrightarrow{P} \in B_k(C(L))$. Denote $q_{l,k} = \sum T^{\omega(\beta) \cap \omega} q_{\beta;l,k}(\overrightarrow{Q}, \overrightarrow{P})$.

We now may define the i-operator (without bulk deformation) using the q-operator as follows (pp. 175 of [18]):

$$i^{\#}_{qm,b}(g) = (-1)^{\deg g} \sum_{\beta \in H_2(M,L;Z)} \sum_{k=1}^{\infty} \rho_{b_0}(\partial\beta) T^{\omega\cap\beta} q_{\beta;1,k}(g;(b^+)^{\otimes k}).$$

We will next explain the $i^{\#}_{qm,b}$-image of units in what follows. This already appeared in the proof of Theorem 23.4 of [23].

Definition 2.9. Let $(A, \mathfrak{m})$ be a filtered A_∞-algebra. $e \in A$ is called a (strict) unit if:

$$\mathfrak{m}_2(e,x) = (-1)^{\deg x}\mathfrak{m}_2(x,e) = x, \ \forall x \in A,$$
$$\mathfrak{m}_k(\cdots, e, \cdots) = 0, \ k \neq 2.$$

There are weaker notions of units of A_∞-algebras. In particular, it was introduced in [18], Section 3.3.1 the notion of homotopy units. We will not reproduce the definition here, but a few words of explanation may be appropriate. Roughly speaking, we say $e \in (A, \mathfrak{m})$ is a *homotopy unit* if there is a canonically associated filtered filtered A_∞-algebra $(A^+, \mathfrak{m}^+)$ which is homotopy equivalent to A with a strict unit e^+. Moreover, e^+ is homotopic to e under the canonical homotopy equivalence from A to A^+.

On the chain level, both $C(M)$ and $C(L)$ can both be extended to a homotopy equivalent cochain complex $C^+(M)$ and $C^+(L)$, while $q_{l,k}$ can also be extended to:

$$q^{++}_{l,k} : E_l C^+(M) \otimes B_k C^+(L) \to C^+(L).$$

Moreover, (3.8.36.1) and (3.8.36.2) shows that

$$q^{++}_{1,0}(\tilde{e}^+, 1) = e^+, \tag{2.1}$$

$$q^{++}_{1,k}(\tilde{e}^+, x) = 0. \tag{2.2}$$

Here $\widetilde{e}^+ \in C^+(M)$ is homotopic to $PD([M]) \in C(M) \subset C^+(M)$, and e^+ is the strict unit in $C^+(L)$ thus homotopic to $PD([L]) \in C(L)$. By a general argument in Section 7.4.2-7.4.6 in [18], all constructions above can be passed to the homology level on the canonical models. Namely, we may define $q^{can}_{\beta;l,k} : E_l H(M;\mathbb{C}) \otimes B_k H(L;\mathbb{C}) \to H(L;\mathbb{C})$, hence also induce an operator $i^*_{qm,b}$ from $i^{\#}_{qm,b}$. As explained in [18], Theorem A, $PD([L])$ is a strict unit in the canonical model. Therefore, discussions above proved that, on the homology level,

$$i^*_{qm,b}(PD([M])) = PD([L]). \tag{2.3}$$

Our proof of Theorem 1.1 mainly relies on the following:

Theorem 2.10 (Theorem 18.8, [23])**.** *Let L be a relatively spin Lagrangian submanifold of M, $b \in \widehat{\mathcal{M}}_{weak}(L, \mathfrak{m})$ be a weak bounding cochain. $e \in QH^*(M;\Lambda)$.*

(1) If $e \cup e = e$ and $i^*_{qm,b}(e) \neq 0$, then L is ζ_e-heavy.

(2) If $QH^*(M;\Lambda) = \Lambda \times Q$ is a direct factor decomposition as a ring, and e comes from a unit of the factor Λ which satisfies $i^*_{qm,b}(e) \neq 0$, then L is ζ_e-superheavy.

Corollary 2.11. *Suppose $QH^*(M;\Lambda) = \bigoplus_{i=1}^n \Lambda e_i$ as a ring, for $e_i \in QH^*(M;\Lambda)$ being a series of idempotents (in particular QH^* is semi-simple). If $HF^*((L,b);\Lambda) \neq 0$, then L is superheavy for certain partial symplectic quasi-state ζ_{e_k}, $1 \leq k \leq n$.*

Proof. This is implicit from the proof of Theorem 23.4, [23]. From (2.3), $i^*_{qm,b}$ sends the unit to the unit. Therefore, at least one of the idempotents e_k has non-vanishing image. From Theorem 2.10, L is superheavy. □

Remark 2.12. Equation (2.3) can also be deduced from the fact that $i^*_{qm,b}$ is indeed a ring homomorphism. See Remark 17.16 [23], Section 31 in [22] and [1].

3. A semi-toric system of $\mathbb{C}P^2$

3.1. **Description of the system.** We recall from [7] the semi-toric system of $\mathbb{C}P^2$ particularly suitable for our problem. We first briefly recall the semi-toric model for $S^2 \times S^2$ following the idea of [16] and [32]. Write $S^2 \times S^2$ as

$$\{x_1^2 + y_1^2 + z_1^2 = 1\} \times \{x_2^2 + y_2^2 + z_2^2 = 1\} \subset \mathbb{R}^3 \times \mathbb{R}^3.$$

Let

$$\widetilde{F}(x_1, y_1, z_1; x_2, y_2, z_2) = z_1 + z_2,$$
$$\widetilde{G}(x_1, y_1, z_1; x_2, y_2, z_2) = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2},$$

then

$$\Phi_{S^2 \times S^2} = (F, G) := (\widetilde{F} + \widetilde{G}, 2 - \widetilde{G}) : S^2 \times S^2 \to \mathbb{R}^2$$

defines a Hamiltonian system. $\widetilde{G}$ is not integrable when it equals 0, that is, at the anti-diagonal. This Hamiltonian system gives a moment polytope as in Figure 1 up to a rescale of the symplectic form, with a singularity at $(0, 1)$ representing a Lagrangian sphere.

Another useful point of view is to consider S^2 equipped with the standard round metric, which induces a metric on its cotangent bundle. $S^2 \times S^2$ is obtained from T^*S^2 by a symplectic cut at the hypersurface

$$M_1 = \{p \in T^*S^2 : |p| = 1\}.$$

The circle action on this hypersurface is exactly the unit-speed geodesic flow we use for cutting. See [27] for details of the construction of symplectic cuts. In this perspective, we may describe Φ in a geometric way. Consider the rotation of S^2 along an axis. The cotagent map of this rotation generates a circle action $\tau_{\widetilde{F}}$ on the whole T^*S^2. Another circle action $\tau_{\widetilde{G}}$ is generated by the unit geodesic flow on the complement of the zero section (we already used it for symplectic cut above). Both $\tau_{\widetilde{F}}$ and $\tau_{\widetilde{G}}$ descend under the symplectic cut thus induces a semi-toric system on $S^2 \times S^2$. It is not hard to check that they are equal to the Hamiltonian actions induced by $\widetilde{F}$ and $\widetilde{G}$ above, which justifies their notations.

We now consider the case for $\mathbb{C}P^2$. There is indeed a semi-toric picture induced from the one for $S^2 \times S^2$. Consider the $\mathbb{Z}_2$-action on T^*S^2 induced by the antipodal map on the zero section, then both $\tau_{\tilde{F}}$ and $\tau_{\tilde{G}}$ are $\mathbb{Z}_2$-equivariant. It is readily seen that, the symplectic cut at the level set M_1 is also $\mathbb{Z}_2$-equivariant. Therefore we have the following commutative diagram, which is equivariant with respect to the action of $\tau_{\tilde{F}}$ and $\tau_{\tilde{G}}$:

$$\begin{tikzcd}
T_1^*S^2 \arrow[r, hook] \arrow[d, "\pi"] & S^2 \times S^2 \arrow[d, "\iota"] \\
T_1^*\mathbb{R}P^2 \arrow[r, hook] & \mathbb{C}P^2
\end{tikzcd}$$

Here π is the $2-1$ cover over $T^*\mathbb{R}P^2$ and ι is the standard two-fold branched cover from the affine quadric to $\mathbb{C}P^2$, branching along the diagonal. It is not hard to see that such a toric degeneration model leads to a degenerated toric polytope as in Figure 2, up to an appropriate rescaling. In this case the line area of $\mathbb{C}P^2$ is 2, since the antipodal map halves the geodesics (if the line class area is 1, the sizes in Figure 2 would have been $\frac{1}{2}$ by 2). We will denote this semi-toric moment map as $\Phi_{\mathbb{C}P^2}$.

3.2. **Symplectic cutting $\mathbb{C}P^2$.** The main ingredient of our proof, following an idea of the arxiv version of [21], is to split $\mathbb{C}P^2$ into two pieces and glue the holomorphic curves. The splitting we use is described as follows. We continue to regard $\mathbb{C}P^2$ as a result of cutting along M_1 in $T^*\mathbb{R}P^2$. Consider $M_\epsilon = \{|p| = \epsilon\} \subset T^*\mathbb{R}P^2 \hookrightarrow \mathbb{C}P^2$. A further symplectic cut along M_ϵ results in two pieces, and we examine this cutting in slightly more detailed.

Let X_0, X_1 be the two components of $\mathbb{C}P^2 \backslash M_\epsilon$, where X_1 contains the original $\mathbb{R}P^2$. Their closure, denoted X_0' and X_1', respectively, has a boundary being the lens space

$L(4,1)$ equipped with the standard contact form (the one coming from S^3 quotiented by a $\mathbb{Z}_4$-action), and therefore a local S^1-action of the neighborhood.

As is constructed in [27], by quotienting such an action on ∂X_1 and gluing back to X_1', one completes the symplectic cutting and this operation results in $X_1'' := (\mathbb{C}P^2, 2\epsilon\omega_0)$. $X_1''\backslash X_1$ is an embedded symplectic divisor in X_1'' of class $2H$, which we called the *cut locus*. The same procedure on the other piece X_0 leads to a minimal symplectic 4-manifold (see for example Lemma 1.1 in [11]), along with a symplectic sphere of self-intersection $(+4)$ inherited from the quadric $Q := \{x^2+y^2+z^2=0\}$ in the original $\mathbb{C}P^2$. Moreover, it contains a symplectic sphere of self-intersection (-4) as cut locus, from which we see X_0'' is indeed the symplectic fourth Hirzebruch surface F_4 from McDuff's famous classification of rational and ruled manifolds [30].

We now want to take a dual point of view of such a cut process. Biran's decomposition theorem for $\mathbb{C}P^2$ ([3]) implies that $\mathbb{C}P^2\backslash\mathbb{R}P^2$ is indeed a symplectic disk bundle $\mathcal{O}(4)$ over a sphere, where the zero section has symplectic area 4, and the symplectic form is given by $\pi^*\omega_\Sigma + d(\bar{r}^2\alpha)$. Here π is the projection to the zero section, ω_Σ a standard symplectic form on the sphere up to a rescale, r the radial coordinate of the fiber and α a connection form of the circle bundle associated to $\mathcal{O}(4)$. Then the fiber class has at most symplectic area 1, and the total space can be identified symplectically with $\mathbb{C}P^2\backslash\mathbb{R}P^2$ with the standard sympletic form.

In this case X_1' is identified with $\{|\bar{r}| \leq 1-\epsilon\} \subset \mathcal{O}(4)$ and the geodesic flow in $T^*\mathbb{R}P^2$ is identified with the action of the one obtained by multiplying $e^{i\theta}$ in each fiber. Therefore, one may perform a symplectic cut along $|\bar{r}| = 1-\epsilon$ for $1 \gg \epsilon > 0$, the resulting manifold is a symplectic F_4, where the form is compatible with the standard (integrable) complex structure obtained as $P(\mathcal{O}\oplus\mathcal{O}(4))$. To summarize, we have the following (see also Figure 3):

Lemma 3.1. *Consider $\mathbb{C}P^2$ as a consequence of symplectic cut along the contact type hypersurface $\{|p| = 1\} \subset T^*\mathbb{R}P^2$. Then a further symplectic cut along $\{|p| = \epsilon\}$ results in a $\mathbb{C}P^2$ with rescaled symplectic form, as well as a symplectic fourth Hirzebruch surface whose zero section has symplectic area equal 2. Moreover, the symplectic $\mathbb{C}P^2$ comes naturally with a symplectic quadric as the cut locus, and F_4 with a (-4)-sphere as cut locus.*

Remark 3.2. Discussions above seem to be well-known. For a dual perspective via symplectic fiber sum, one is referred to for example [11]. In particular, the above cutting can be seen as a reverse procedure of symplectic rational blow-down of the (-4)-sphere in the symplectic F_4.

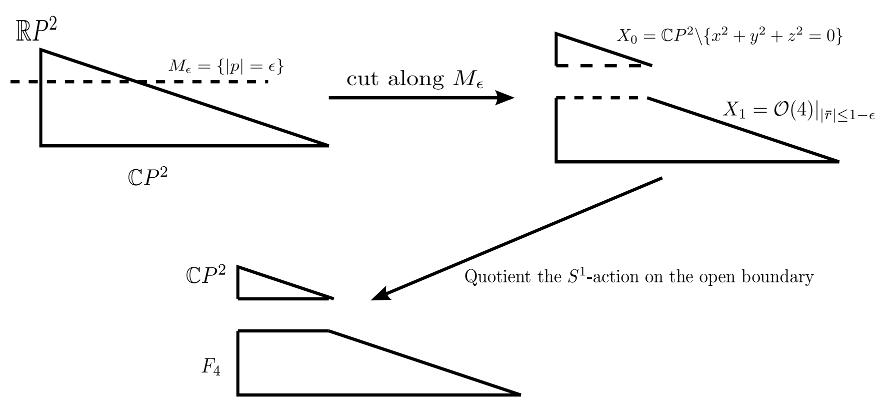

FIGURE 3. Cutting along M_ϵ

3.3. **Second homology classes of $\mathbb{C}P^2$ with boundary on a toric fiber.** Following the Lagrangian Floer theory set up by Fukaya-Oh-Ohta-Ono [18], we will classify pseudo-holomorphic disks of Maslov index 2. For this we first need to understand the homology classes in $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ when L is a Lagrangian fiber, as well as their Maslov indices.

From the usual long exact sequence for relative homology, one easily sees that $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ has dimension 3. To describe these classes in a convenient way for our applicaitons, it is useful to pass to F_4, which is considered as a result of symplectic cut as in last section.

In our applications, we always assume that the Lagrangian toric fibers of which we concern lies in the resulting F_4. Take the standard toric complex structure J_0 in F_4 as in [10]. From their classification, one obtains eight homology classes of interests, marked as $[D_i]$ and $[e_i]$, $i = 1, 2, 3, 4$ as Figure 4 below. In the figure, e_i are the equivariant divisors and, as relative cycles, D_i denote the image of J_0-holomorphic disks which intersects e_j exactly δ_{ij} times counting multiplicity. For ease of drawing we did not draw D_3 perpendicular to e_3, but it is understood in the way of how Cho and Oh described.

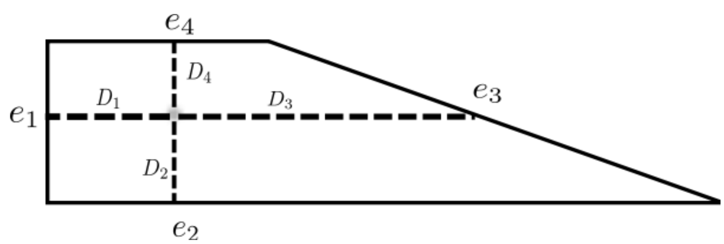

FIGURE 4. $H_2(F_4, L; \mathbb{Z})$

Out of these eight classes, one has a basis of $H_2(F_4, L; \mathbb{Z})$ consisting of $[e_1]$, $[e_2]$, $[D_1]$ and $[D_2]$. Other classes have relations

$$[e_3] = [e_1], \quad [e_4] = [e_2] - 4[e_1], \tag{3.1}$$

$$[D_1] + [D_3] + 4[D_2] = [e_2], \quad [D_2] + [D_4] = [e_1]. \tag{3.2}$$

Now notice that there is a natural homomorphism by restriction:

$$\iota : H_2(F_4, L; \mathbb{Z}) \to H_2^{Moore}(F_4 \backslash e_4, L; \mathbb{Z}).$$

ι is surjective with kernel $[e_4]$, and the classes in $H_2^{Moore}(F_4 \backslash e_4, L; \mathbb{Z})$ can still be represented by e_i, $i = 1, 2, 3$ and D_j, $j = 1, 2, 3, 4$ appropriately punctured with the same relations as in (3.1), (3.2). These facts can be easily seen from the duality between the Borel-Moore homology and the usual homology.

On the other side of the cutting, which is $\mathbb{C}P^2 \backslash Q$, where $Q = \{x^2 + y^2 + z^2 = 0\}$ being the standard quadric, the second Borel-Moore homology contains only a 2-torsion. Such a torsion is represented by cycles $C \backslash Q$, when C is a cycle in $\mathbb{C}P^2$ of class H intersecting Q transversely. Here, H denotes the line class of $\mathbb{C}P^2$. However, it is still beneficial to distinguish these cycles in a more refined way later. In $\mathbb{C}P^2 \backslash Q$, we will only consider Borel-Moore cycles with asymptotics equal the union of certain Reeb orbits of $\partial^\infty(\mathbb{C}P^2 \backslash Q) = L(4, 1)$. Therefore, we will denote by kH' being the *equivalence classes of cycles which descend to a cycle representing $kH \in H_2(\mathbb{C}P^2, \mathbb{Z})$ in the standard compactification.* Here the standard compactification means quotienting the S^1-action on the boundary, in other words, completing the symplectic cut.

Relations between classes in $X_0 = F_4 \backslash e_4$ and $X_1 = \mathbb{C}P^2 \backslash Q$: To describe the relations between classes in the two pieces, we first fix a basis of $H_2^{Moore}(F_4 \backslash S_\infty, L; \mathbb{Z})$ consisting of $\{\iota[e_2], \iota[D_1], \iota[D_2], \iota[D_4]\}$. When no possible confusion occurs, we will simply suppress ι. Notice that cycles in kH' in X_1 has $2k$ punctures counting multiplicity, which matches with cycles with coefficient $2k$ in the D_4-component in X_0. Of particular interests, by matching

a cycle $C_H \subset X_1$ in class H' with two cycles of class $2[D_4]$ with correct asymptotics, one obtains a relative cycle in $\mathbb{C}P^2$ with boundary on L. The class in $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ represented by such a cycle is denoted as $[D_4'] = 2[D_4]\#[H']$.

To understand $[D_4']$ more explicitly, notice that $\partial[D_4'] = 2\partial[D_4] = -2\partial[D_2] \in H_1(L; \mathbb{Z})$. Therefore, one may match a cycle in class $[D_4']$ with one in $2[D_2]$ to obtain a closed cycle in $\mathbb{C}P^2$. Such a cycle intersects e_2 positively twice counting multiplicities, and therefore represents nothing but class $H \in H_2(\mathbb{C}P^2; \mathbb{Z})$. In summary, we deduced that:

$$[H']\#2[D_4\#2[D_2] = [D_4']\#2[D_2] = H \in H_2(\mathbb{C}P^2; \mathbb{Z}) \tag{3.3}$$

Classes $[e_1]$ and $[e_3]$ does not extend naturally to closed classes as in $H_2(\mathbb{C}P^2)$. However, as what we did to $[D_4]$, twice of them caps cycles in H' of X_1. Therefore we also have:

$$[H']\#2[e_1] = [H']\#2[e_3] = H \in H_2(\mathbb{C}P^2; \mathbb{Z}) \tag{3.4}$$

It is now readily seen that $\{H, [D_1], [D_2]\}$ forms a basis of $H_2(\mathbb{C}P^2, L; \mathbb{Z})$, where $F_4 \backslash e_4$ is (symplectically) embedded to $\mathbb{C}P^2$ in a canonical way, thus induces a natural inclusion of Borel-Moore two cycles.

3.4. **Computation of the relative Chern numbers and Conley-Zehnder indices.** Since our will need to classify the contributions of all Maslov two holomorphic disks in $\mathbb{C}P^2$ to our potential function, we need to find the classes with Maslov two. A technical reason which makes our case slightly more complicated than the case of a Lagrangian S^2 is that there is no natural splitting of $T(T^*\mathbb{R}P^2)$. This is caused by the non-orientability of $\mathbb{R}P^2$. In this case the first Chern number cannot be "pushed" into the complement of a Weinstein neighborhood of $\mathbb{R}P^2$. However, we will use a trivialization of the splitting surface M_ϵ, which seems even more natural and convenient in the toric context.

As we already saw, there is an S^1-action on $\partial X_i' = M_\epsilon$ for both $i = 0, 1$. In the toric picture of $\mathbb{C}P^2 \backslash \mathbb{R}P^2$, such an S^1-action induces a vector field on M_ϵ which is dual to $\frac{\partial}{\partial x_2}$ in the moment polytope. This action induces a natural trivialization of the contact distribution over its own orbits. We will call such a trivialization Φ and use it to compute the Conley-Zehnder and first Chern numbers. For the definitions of these two invariants one is referred to [12], or [17, 24]

By definition, the Poincare return map with respect to such a trivialization is always identity, therefore,

$$\mu_{CZ}^{\Phi} \equiv 0. \tag{3.5}$$

We will pursue the first Chern number for (Borel-Moore) classes described in Section 3.3 in the rest of the section.

We start with X_0. As always we assume the Lagrangian torus fiber is contained in this side. Consider again the $\mathcal{O}(4)$ disk bundle as in 3.2, which we cut along the hypersurface $M_{\epsilon/2}\{r = 1 - \frac{\epsilon}{2}\}$ to obtain a symplectic fourth Hirzebruch surface $\overline{X}_0$. One may also equip the compatible toric complex structure. The anti-canonical divisor is defined by the equivariant divisors on the boundary of the moment polytope, therefore, the anti-canonical line bundle $\bigwedge^2 T\overline{X}_0$ admits an equivariant section ξ vanishing exactly on the boundary equivariant divisors with order 1.

Embed X_0 equivariantly into $\overline{X}_0$. Take any cycle $u : \Sigma \to X_0$ with boundary on a torus fiber L and asymptotics being orbits of ∂X_0. It has boundary Maslov index zero if we take the trivialization induced by the torus action near L. Assume that u intersects transversally with the equivariant divisors. The pull-back $u^* \bigwedge^2(TX_0, J)$ thus comes naturally with a section $u^*\xi$ which vanishes at $u(\Sigma) \cap \bigcup_{i=1}^4 e_i$ with order ± 1 depending on the intersection form. $u^*\xi$ is clearly equivariant with the S^1-action on ∂X_0 and the torus boundary thus agreeing with the trivialization there. This observation computes immediately the following:

$$c_1^\Phi(D_1) = c_1^\Phi(D_2) = c_1^\Phi(D_3) = 1, \quad c_1^\Phi(D_4) = 0. \tag{3.6}$$

Notice also that the first chern number of e_2 is independent of the choice of trivializations. From (3.1) and (3.2) we may compute the rest of the chern numbers summarized as follows:

$$c_1^\Phi(e_1) = c_1^\Phi(e_3) = 1, \quad c_1^\Phi(e_2) = 6. \tag{3.7}$$

For the relative Chern classes in X_1, we again focus on cycles with asymptotics equal copies of S^1-orbits on M_ϵ. Note that, when counting multiplicity, there are always even number of S^1-orbits since simple orbits represents a non-trivial element in $\pi_1(T^*\mathbb{R}P^2)$. The class kH' has $2k$ asymptotics, which can be capped by $2k[D_4]\#2k[D_2]$ to become a closed cycle in $\mathbb{C}P^2$ from 3.2. Such a class intersects positively with e_2 at $2k$ points, thus itself being the class kH in $\mathbb{C}P^2$. From our computation in X_0, we see that

$$c_1^\Phi(kH') = 3k - 2k = k > 0. \tag{3.8}$$

4. Classification of Maslov 2 disks

4.1. **Independence of the choice of almost complex structure.** We recall from the Appendix of [21] regarding the dependence of Floer cohomology on the almost complex structures. In general one can make Floer theory of a Lagrangian independent of the choice of almost complex structure in the sense of filtered A_∞-algebras, but this usually requires techniques of virtual perturbations, and the effects on the Floer cohomology is not as transparent. See [18] for example.

If one prefers working with geometric situations and hope for independence of almost complex structures in a less sophisticated way, some extra assumptions are required. We borrowed the following from [21]:

Assumption: If $\beta \in \pi_2(M, L; J)$ is non-zero and $\mathcal{M}_{k+1}(\beta) \neq \emptyset$, then $\mu_L(\beta) \geq 2$.

Under this assumption, no virtual components enters the computation. In our case of $\dim L = 2$, this assumption is satisfied when the almost complex structure is generic. In particular we have:

Lemma 4.1. *$HF^*(\mathbb{C}P^2, L(u); \Lambda_{0,nov}; J)$ is independent of the choice of generic J.*

In this case, one can construct a canonical model $(H(L;\Lambda_{0,nov}), \mathfrak{m}^{can})$ of filtered A_∞ algebras, so that the deformed zeroth differential is written as:

$$\mathfrak{m}_0^{can,b}(1) = \sum_{\mu_L(\beta)=2} eT^{\omega(\beta)} ev_{0*}([\mathcal{M}_1(\beta)]),$$

therefore,

$$\mathfrak{m}_0^{can,b}(1) = \sum_{\mu_L(\beta)=2} eT^{\omega(\beta)} exp(b \cap \partial\beta) ev_{0*}([\mathcal{M}_1(\beta)]),$$

and we can compute the potential function by

$$PD^L(b) = \mathfrak{m}_0^{can,b}(1) \cap [L]/e. \tag{4.1}$$

Here $b \in H^1(L; \Lambda_0)$ is a solution of the Maurer-Cartan equation. Therefore, we will focus on the computation of $ev_{0*}([\mathcal{M}_1(\beta)])$ in the rest of the section.

4.2. **A quick review on SFT and neck-stretching.** In this section we collect basic definitions and facts from symplectic field theory, especially the part of neck-stretching, mostly for readers' convenience and to fix notations. For more details, we refer interested readers to [12], [5], and other expositions such as [17, 24, 29].

Given a closed symplectic manifold (M, ω), we call (H, α) a *contact type hypersurface* if there is a neighborhood V of H, such that V is diffeomorphic to $(-\epsilon, \epsilon) \times H$, and ∂_s is a Liouville vector field in V, that is, $\mathcal{L}_{\partial_s}\omega = \omega$. Here s is the coordinate of the first

component of U. In this case, $\alpha = i_{\partial_s}\omega$ is a contact form, of which the contact distribution is denoted ξ, and the Reeb flow denoted by R.

An almost complex structure $J \in \mathcal{J}_\omega$ is called *adjusted* if the following conditions hold in U:

(i) $J|_\xi = \widetilde{J}$ is independent of s;
(ii) $J(\partial_s) = R$.

We now consider a deformation of a given adjusted almost complex structure J. Let $V_t = [-t-\epsilon, t+\epsilon]$ and $\beta_t : V_t \to [-\epsilon, \epsilon]$ be a strictly increasing function with $\beta_t(s) = s+t$ on $[-t-\epsilon, -t-\epsilon/2]$ and $\beta_t(s) = s-t$ on $[t+\epsilon/2, t+\epsilon]$. Define a smooth embedding $f_t : V_t \times H \hookrightarrow M$ by:

$$f_t(s, m) = (\beta_t(s), m).$$

Let $\bar{J}_t$ be the $\frac{\partial}{\partial s}$−invariant almost complex structure on $V_t \times H$ such that $\bar{J}_t(\frac{\partial}{\partial s}) = R$ and $\bar{J}_t|_\xi = J|_\xi$. Glue the almost complex manifold $(M \backslash f_t(V_t \times H), J)$ to $(V_t \times H, \bar{J}_t)$ via f_t to obtain the family of almost complex structures J_t on M.

Notice that each J_t agrees with J away from the collar $(-\epsilon, \epsilon) \times H$. And on this collar, it agrees with J on ξ. Suppose H is separating, denote $M \backslash H = W \cup U$, where W has a concave boundary and U a convex boundary. When $i \to \infty$ the neck-stretching process results in an almost complex structure J_∞ on the union of symplectic completions $\overline{W} = (-\infty, 0] \times H \cup W$ of W and $\overline{U} = U \cup [0, +\infty)$ of U. On the cylindrical ends, we require $J|_\infty(\partial_s) = R$ and $J|_\infty = J|_\xi$ similar to the definition of $\overline{J}_t$. In an exact same way, we define J_∞ on $SH = ((-\infty, +\infty) \times H, d(e^t\alpha))$, the symplectization of H.

Let $M_\infty = \overline{W} \cup SH \cup \overline{U}$, and J_∞ be the almost complex structure defined above. Let Σ be a Riemann surface with nodes. A *level-k holomorphic building* consists of the following data:

(i) (level) A labelling of the components of $\Sigma \backslash \{\text{nodes}\}$ by integers $\{1, \cdots, k\}$ which are the *levels*. Two components sharing a node differ at most by 1 in levels. Let Σ_r be the union of the components of $\Sigma \backslash \{\text{nodes}\}$ with label r.
(ii) (asymptotic matching) Finite energy holomorphic curves $v_1 : \Sigma_1 \to U$, $v_r : \Sigma_r \to SH$, $2 \le r \le k-1$, $v_k : \Sigma_k \to W$. Any node shared by Σ_l and Σ_{l+1} for $1 \le l \le k-1$ is a positive puncture for v_l and a negative puncture for v_{l+1} asymptotic to the same Reeb orbit γ. v_l should also extend continuously across each node within Σ_l.

Now for a given stretching family $\{J_{t_i}\}$ as previously described, as well as J_{t_i}-curves $u_i : S \to (M, J_{t_i})$, we define the Gromov-Hofer convergence as follows:

A sequence of J_{t_i}-curves $u_i : S \to (M, J_{t_i})$ is said to be *convergent to a level-k holomorphic building* v in Gromov-Hofer's sense, using the above notations, if there is a sequence of maps $\phi_i : S \to \Sigma$, and for each i, there is a sequence of $k-2$ real numbers t_i^r, $r = 2, \cdots, k-1$, such that:

(i) (domain) ϕ_i are locally biholomorphic except that they may collapse circles in S to nodes of Σ,
(ii) (map) the sequences $u_i \circ \phi_i^{-1} : \Sigma_1 \to U$, $u_i \circ \phi_i^{-1} + t_i^r : \Sigma_r \to SH$, $2 \le r \le k-1$, and $u_i \circ \phi_i^{-1} : \Sigma_k \to W$ converge in C^∞-topology to corresponding maps v_r on compact sets of Σ_r.

Now the celebrated compactness result in SFT reads:

Theorem 4.2 ([5])**.** *If u_i has a fixed homology class, there is a subsequence t_{i_m} of t_i such that $u_{t_{i_m}}$ converges to a level-k holomorphic building in the Gromov-Hofer's sense.*

Definitions and statements above holds true for bordered stable maps with no extra complications, as long as the Lagrangian boundary does not intersect the contact type boundary H. Since the choice of almost complex structure will play an important role in subsequent sections, we would like to specify a special class of adjusted almost complex structures for later applications.

Denote e_1', e_2' and e_3' as the pre-images of the three edges of $\Phi_{\mathbb{C}P^2}$, numbering in a coherent way as in F_4 in Section 3.3.

Definition 4.3. We say $J \in \mathcal{J}^{\epsilon}_{tadj}$, the space of *compatible toric adjusted almost complex structures*, if J is compactible with $\omega_{std}|_{X_0}$, and adjusted to the hypersurface $M_\epsilon = \Phi^{-1}_{\mathbb{C}P^2}(\{x_2 = 1-\epsilon\})$, while e_1', e_2' and e_3' are J-holomorphic in $X_0 = \Phi^{-1}_{\mathbb{C}P^2}(\{x_2 < 1-\epsilon\})$. Moreover, J is invariant under the circle action generated by Reeb flow in a neighborhood of M_ϵ.

It is not hard to see that $\mathcal{J}^{\epsilon}_{tadj}$ is non-empty. Notice e_1', e_2' intersects M_ϵ transversely, and are foliated by simple orbits of the circle action. Moreover, the Liourville vector field near M_ϵ is invariant under the circle action, and is tangent to e_1' and e_3' in a neighborhood of M_ϵ. Therefore, one only needs to define the almost complex structure to be adjusted, whose restriction to the contact distribution is invariant under the circle action, then extend to the rest of X_0 in an $\omega_{std}|_{X_0}$-compatible way so that $e_i' \cap X_0$ are holomorphic for $i = 1, 2, 3$.

We would like to point out that, one can still achieve transversality within $\mathcal{J}^{\epsilon}_{tadj}$ because no holomorphic curves lies entirely in the region we fixed the almost complex structure, with the exceptions of e_i', which are clearly regular on their own right (see Wendl's automatic transversality in Section 4.3.3). Moreover, the space of such almost complex structures is contractible, because it is just the space of sections of a bundle with contractible fibers with prescribed values on a closed set.

4.3. **Contributions of holomorphic disks of Maslov index 2.** In this section we will compute terms involved in 4.1 by studying evaluation of several moduli spaces. Since we are in a non-toric situation, we will apply the techniques of symplectic field theory. We will first study the configurations of limits under neck-stretching of holomorphic disks of Maslov index 2, then study all possible cases.

Here we fix some more notations convenient for our exposition. Given a symplectic manifold (M, ω) (closed or with cylindrical boundary) equipped with a compatible almost complex structure J (adjusted to the cylindrical boundary, if exists). In this case, let $L \subset M$ be a closed Lagrangian submanifold and $\beta \in H_2(M, L; \mathbb{Z})$, the moduli space of J-holomorphic disks with a boundary marked point in class β is still denoted by $\mathcal{M}_1(\beta; M, J)$.

For a Borel-Moore class B, we also consider the moduli space of holomorphic disks punctured at an interior point, and with one marked point on the boundary, which we denote as $\mathcal{M}_1^k(B; M, J)$ if the interior puncture is asymptotic to k times of a simple Reeb orbit.

Finally, let $\overrightarrow{k} = (k_1, k_2, \cdots, k_i)$ be an i-vector, then $\mathcal{M}^{\overrightarrow{k}}(B; M, J)$ is the moduli space of i^{th}-punctured holomorphic spheres with k_l times of a simple Reeb orbit at the l^{th} interior puncture. We also consider the evaluation maps:

$$ev^i : \mathcal{M}^{\overrightarrow{k}}(B; M, J) \to N$$

where N is the Morse-Bott manifold where the i^{th} puncture lies in. When no confusion is likely to occur, we sometimes suppress M and J.

4.3.1. *Neck-stretching of holomorphic disks.* Given $J \in \mathcal{J}^{\epsilon}_{tadj}$, we may perform neck-stretching described in 4.2, and denote $J^+ := J_\infty|_{X_0}$. Recall that X_0 can be compactified to F_4 by collapsing the circle action on the boundary. Under this operation, the asymptotic boundary of X_0 collapses to the edge e_4, and (part of) e_i' gives rise to e_i in F_4 for $i = 1, 2, 3$. Given a J^+-holomorphic punctured curve C of finite energy with boundary on L (possibly empty), from the asymptotic analysis in [4], C can also be compactified to a well-defined 2-cycle $\overline{C} \in C_2(F_4, L; \mathbb{Z})$ with $\overline{C} \cap e_4 \geq 0$. Also $\overline{C} \cap e_i \geq 0$, $i = 1, 2, 3$, following the positivity of intersection and the definition of $\mathcal{J}^{\epsilon}_{tadj}$. Notice that $[D_i]$, $i = 1, 2, 3, 4$ also forms a basis of $H_2(F_4, L; \mathbb{Z})$ from (3.1)(3.2), and by Poincare duality, elements in $H_2(F_4, L; \mathbb{Z})$ can be identified by their pairings with divisors e_i for $i = 1, 2, 3, 4$. We therefore proved:

Lemma 4.4. *For $J \in \mathcal{J}^{\epsilon}_{tadj}$, a J^+-holomorphic 2-cycle C has its class in the positive span of $\{[D_i]\}_{i=1}^4$. In particular, the Maslov index $\mu^\Phi(C) \geq 0$, and equality holds if and only if $[C] = k[D_4]$ for some $k \in \mathbb{Z}^{\geq 0}$.*

We are now ready to prove:

Lemma 4.5. *For $J \in \mathcal{J}^{\epsilon}_{tadj}$, the equation (4.1) has at most four terms of contributions coming from $[D_1]$, $[D_2]$, $[D_3]$ and $[D_4']$.*

Proof. Given $J \in \mathcal{J}^{\epsilon}_{tadj}$, by neck-strething we obtain a family of almost complex structure J_t. Given a homology class A which admits J_{t_i}-holomorphic disks with Maslov index 2 for a sequence $t_i \nearrow \infty$, $i \in \mathbb{Z}^+$. By the compactness theorem 4.2, it converges to a holomorphic building. We then have one of the following cases:

Case 1*: the X_1-part of the holomorphic building is empty.*
In this case, X_0 is symplectomorphic to $F_4 \backslash e_4$. Therefore, (3.6) and Lemma 4.4 implies D_1, D_2, D_3 are the only possibilities, otherwise the Maslov index must exceed 2.

Case 2*: the X_1-part of the holomorphic building is non-empty.*
Consider the X_1-part of the holomorphic building S_1. Since it must have periodic orbits as asymptotes, it is a Borel-Moore cycle of class kH' for some $k \in \mathbb{Z}^+$. Therefore, $c_1^{\Phi}(S_1) \geq 1$ by (3.8). To close up this cycle in $\mathbb{C}P^2$, one must cap S_1 by some cycle in X_0. However, from our computations in Section 3.4 and Lemma 4.4 we saw that all holomorphic cycles but those with classes in multiples of $[D_4]$ have positive first Chern number. Therefore, the only J_∞-holomorphic building with Maslov index 2 consists of a cycle in class H' in X_1 and a holomorphic disk in the class $2[D_4]$ in X_0. The class they form in $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ is $[D_4']$.

□

Lemma 4.5 narrows our study down to four classes. Notice the above two lemmata assumes no genericity of J. Moreover, from the proof we see that to understand the contributions of $[D_i]$, $i = 1, 2, 3$, it suffices to study the stretching limit. To understand holomorphic disks in $[D_4']$, we need a slightly more detailed description of the limit holomorphic building:

Lemma 4.6. *When $t \to \infty$, J_t-holomorphic disks in class D_4' converge to holomorphic building consisting of the following levels:*

(1) the X_1-part is holomorphic plane in class H' with one asymptotic puncture of multiplicity 2*;*
(2) the symplectization part is a trivial cylinder with one asymptotic puncture of multiplicity 2 *on both positive and negative sides;*
(3) the X_0-part is a holomorphic disk in class $2[D_4]$ *with a single puncture of multiplicity* 2.

Proof. In the proof of Lemma 4.5 we already saw that X_1-part can only be of class H' and X_0-part is a cycle in class $2[D_4]$ by counting Maslov indices and numbers of punctures. To see that X_0-part is a holomorphic disk with a single puncture of multiplicity 2 instead of two simple punctures, notice otherwise the holomorphic building will be forced to have at least genus 1, since simple orbits cannot be capped by disks on X_1 side. This verifies (3).

In the symplectization part, since all orbits have the same period, and the positive end has exactly one orbit of multiplicity 2, the negative end also has at most 2 orbits counting multiplicities. Again since simple orbits do not close up in X_1, there must be two negative ends counting multiplicity. Since the λ-energy is now zero for the symplectization part, the image of the symplectization part is a trivial cylinder. Since branched covers over the trivial cylinder always create genus in this holomorphic building, we conclude that the symplectization part is indeed an unbranched double cover of the trivial cylinder. This verifies (2), as well as that X_1-part has exactly one puncture of multiplicity 2. The rest of assertions in (1) is easy.

□

4.3.2. *Contribution of* $[D_i]$. In this section, we prove that:

Proposition 4.7. *For generic $J \in \mathcal{J}^{\epsilon}_{tadj}$, $deg(ev_{0*}[\mathcal{M}_1([D_i]; J)]) = 1$.*

Proof. From Lemma 4.1, we may perform a neck-stretch on J, so that all disks of $\mathcal{M}_1([D_i]; J_t)$ lie entirely in X_0. Since J is cylindrical near M_ϵ, we have the following claim:

Lemma 4.8. *$(X_0, J^+ = J_\infty|_{X_0})$ is biholomorphic to an open set U of a closed symplectic manifold with a compatible almost complex structure $(\widetilde{X}_0, \widetilde{\omega}, \widetilde{J})$, so that the following holds:*

(i) $(\widetilde{X}_0, \widetilde{\omega})$ is a symplectic F_4;
(ii) $\Sigma = \widetilde{X}_0 \backslash U$ is a $\widetilde{J}$-divisor.

This is simply a translation between the set-up of relative invariants of [28] and the one of symplectic field theory in the case when Reeb orbits foliates the contact type hypersurface. $\widetilde{X}_0$ as a symplectic manifold comes from collapsing the circle action on ∂X_0, which forms a symplectic divisor Σ. Near Σ the symplectic form of $\widetilde{X}_0$ can be written as:

$$\omega = \pi^* \tau_0 + d(r'\lambda'), \tag{4.2}$$

for $\delta > r' > 0$. Here τ_0 is a symplectic form on Σ, r' a radial coordinate; π is the radial projection to Σ, and λ' a connection 1-form (in our case it is also a contact form) on level sets of r', satisfying $d\lambda' = \pi^*\tau_0$. Given any complex structure J on Σ, J can be lifted to the horizontal distributions ξ (i.e. the contact distributions), while the almost complex structure on the whole neighborhood can be defined by further requiring $J(r'\partial_{r'}) = R'$. Here R' is the Hamiltonian flow generated by the local (in our case also global) S^1-action. Conversely, given an almost complex structure satisfying $J(r'\partial_{r'}) = R'$ and invariant under the circle action on $\widetilde{U}\backslash\Sigma$ where $\widetilde{U}$ is a neigbhorhood of Σ, it has a natural extension to Σ.

On the SFT side, endow a symplectic form written as $d(r\lambda)$ to the collar of $H = \partial X_0$, $1 + \delta \geq r > 1$, where λ is the contact form on H. This coordinate can be transformed back to the one in Section 4.2 by taking a *log*-function on the cylindrical coordinate. The zero level-set there becomes the level set $r = 1$ in the current coordinate. In this coordinate, the toric adjustedness of J^+ is equivalent to the invariance under both flows of $r\partial_r$ and R, and that $J^+(\partial_r') = R$, where R are the contact distribution and the Reeb flow, respectively.

Notice the fact that $(H \times (1, 1+\delta), d(r\lambda))$ is symplectomorphic to $(H \times (0, \delta), d\lambda + d(r\lambda))$ just by shifting the r-coordinate. The symplectic cut, in perspective of this coordinate change, is simply to add a divisor Σ to the latter model. In particular, such a shift provides a symplectic identification of a collar neighborhood of ∂X_0 and $\widetilde{U}\backslash\Sigma$. Under such an identification, J^+ induces an almost complex structure $\widetilde{J}$ on $\widetilde{U}\backslash\Sigma$, which is invariant under $r'\partial_{r'}$ and the Hamiltonian flow R' by the assumption of toric adjustedness. It is then straightforward to see that that J^+ extends to the cut divisor Σ in the new coordinate. Extend the identification on $\widetilde{U}$ to a diffeomorphism between $U = \widetilde{X}_0\backslash\Sigma$ and X_0, we induce $\widetilde{J}$ by J^+ on the whole $\widetilde{X}_0$.

Given Lemma 4.8, we may consider the problem in $\widetilde{X}_0$ which is symplectomorphic to F_4, endowed with a compatible pair of almost complex structure $\widetilde{J}$ and ω. By abuse of notations, we denote the corresponding classes after the compactification still as $[D_i]$ and equivariant divisors as e_i, to be compatible with conventions in F_4. A problem arises after the compactification: $\widetilde{J}$ is never generic, in the sense that e_4 has negative chern number, yet always $\widetilde{J}$-holomorphic. However, we can still prove:

Lemma 4.9. *When t is large enough, curves in the compactification of $\mathcal{M}_1([D_i]; \widetilde{X}_0, \widetilde{J})$ are irreducible. Hence*

$$ev_{0*}([\mathcal{M}_1([D_i]; \mathbb{C}P^2, J_t)]) = ev_{0*}([\mathcal{M}_1([D_i]; \widetilde{X}_0, \widetilde{J})]).$$

Proof. We may consider the problem in the limit and replace the left hand side by X_0 and J^+, by Lemma 4.5 (or rather its proof). Further Lemma 4.8 reduces our problem to proving that the contribution of holomorphic disks in $(\widetilde{X}_0, \widetilde{J})$ disjoint from e_4 is sufficient to compute right hand side.

Take $[D_1]$ as an example, and the rest of the cases are similar. Assume $u : \Sigma \to (\widetilde{X}_0, \widetilde{J})$ is a stable curve in the moduli space of right hand side with irreducible components $\Sigma_1, \dots, \Sigma_k$, and the homology classes of all of these components are written in terms of basis $\{[D_i]\}_{i=1}^4$. Let $\Sigma_1 \dots, \Sigma_l$ be components of e_4, while $[e_4] = [D_1] + [D_3] - 4[D_4]$. Now Lemma 4.4 implies that $[\Sigma_i]$, $i > l$ all lie in the positive cone of spanned by $[D_i]$ for $i = 1, 2, 3, 4$. By comparing coefficients of $[D_3]$, we conclude that $l = 0$. It then follows easily that $k = 1$ and $[\Sigma_1] = [D_1]$. Since $[D_1]$ pairs trivially with $[e_4]$, our claim is proved by positivity of intersections.

□

We already saw from Lemma 4.9 that the compactification of $\mathcal{M}_1([D_i]; \widetilde{X}_0, \widetilde{J})$ only contains degenerations of at least codimension 2. Therefore, the standard genericity and cobordism arguments apply. In particular, one may choose a generic path $\{J_t\}_{t\in[0,1]}$ connecting J_0 and $J_1 = \widetilde{J}$ for J_0 also satisfying that e_i, $i = 1, 2, 3, 4$ are J_0-holomorphic. Recall from [10] that there is an integrable complex structure J_0 where $ev_{0*}([\mathcal{M}_1([D_i]; \widetilde{X}_0, J_0)])$ is known to be $[L]$, hence concluding our proof of Proposition 4.7.

□

4.3.3. *The contribution of* $[D_4']$. Our goal of this section is to prove:

Proposition 4.10. *For generic choice of* $J \in \mathcal{J}^{\epsilon}_{tadj}$,

$$deg(ev_{0*}[\overline{\mathcal{M}}_1(D_4'; \mathbb{C}P^2, J)]) = 2.$$

As already explained in previous sections, we will only need to consider $J \in \mathcal{J}^{\epsilon}_{tadj}$ and its neck stretched sufficiently long. For this we first briefly review Wendl's automatic transversality theorem.

One of the new ingredients of Wendl's theorem is the introduction of the invariant *parity*, defined in [26], to the formula. Let Y be a symplectic cobordism, where $Y^{\pm}$ are the positive (resp. negative) boundaries. Given a T-periodic orbit γ of $Y^{\pm}$, one has an associated asymptotic operator, which takes the form of $\mathbb{A} = -I_0\partial_t - S(t)$ on $L^2(S^1, \mathbb{R}^2)$ by taking a trivialization of the normal bundle. Here I_0 is the standard complex structure on $\mathbb{R}^2$, while $S(t)$ is a continuous family of symmetric matrices. For $\lambda \in \sigma(A)$, one may define a *winding number* $w(\lambda)$ to be the winding number of nontrivial λ-eigenfunction of $\mathbb{A}$. It is proved in [26] that $w(\lambda)$ is an increasing function of λ which takes every integer value exactly twice. For non-degenerate operators $\mathbb{A}$ (i.e. $0 \notin \sigma(\mathbb{A})$), we define

$$\alpha_+(\mathbb{A}) = max\{w(\lambda) | \lambda \in \sigma(\mathbb{A}), \lambda < 0\},$$
$$\alpha_-(\mathbb{A}) = min\{w(\lambda) | \lambda \in \sigma(\mathbb{A}), \lambda > 0\}.$$

and $p(\mathbb{A}) = \alpha_+(\mathbb{A}) - \alpha_-(\mathbb{A}) (\text{mod } 2)$. If $\mathbb{A}$ is degenerate, we define $\alpha_{\pm}(\mathbb{A} \pm \delta)$ and $p(\mathbb{A} \pm \delta)$ for small $\delta > 0$. For a given puncture, the actual perturbation depends on which of $Y^{\pm}$ it lies on, as well as whether the moduli space we consider constrains the puncture inside a Morse-Bott family. Chris Wendl pointed out to the author that, in our case when the contact type boundary is foliated by a 2 dimensional family of Reeb orbits, since the eigenvalue 0 has multiplicity 2, *either way of perturbation incurs odd parity.*

Now given a non-constant punctured holomorphic curve $u : \Sigma_g \to Y$, the virtual index is computed as:

$$ind(u) = (n-3)\chi(\Sigma) + 2c_1(u) + \sum_{\gamma^+} (\mu_{CZ}(\gamma^+) + \frac{1}{2}dim(N)) - \sum_{\gamma^-} (\mu_{CZ}(\gamma^-) - \frac{1}{2}dim(N)),$$

Here $\gamma^{\pm}$ runs over all positive (resp. negative) punctures, and N is the Morse-Bott manifold formed by the Reeb orbits. We now define the *normal chern number* as:

$$2c_N(u) = ind(u) - 2 + 2g + \#\Gamma_0 + \#\pi_0(\partial\Sigma_g).$$

Here, Γ_0 denotes the number of punctures of even parities. Having understood these, Wendl's automatic transversality theorem reads:

Theorem 4.11 ([33]). *Suppose* $\dim Y = 4$ *and* $u : (\Sigma, j) \to (Y, J)$ *is a non-constant curve with only Morse-Bott asymptotic orbits. If*

$$ind(u) > c_N(u) + Z(du),$$

then u is regular.

For holomorphic disks in class D_4', we consider the gluing problem of $\mathcal{M}^2(H'; X_1, J_\infty)$ and $\mathcal{M}^2_1(2[D_4]; X_0, J_\infty)$. Note that this is sufficient by the configuration analysis of the limit holomorphic building in Lemma 4.6. The standard gluing argument requires the following conditions:

(1) Curves in both $\mathcal{M}^2(H'; X_1, J_\infty)$ and $\mathcal{M}^2_1(2[D_4]; X_0, J_\infty)$ are regular;
(2) $ev^1 \times ev^1 : \mathcal{M}^2(H'; X_1, J_\infty) \times \mathcal{M}^2_1(2[D_4]; X_0, J_\infty) \to S^2 \times S^2$ is transversal to the diagonal $\Delta \subset S^2 \times S^2$. Here S^2 is exactly the Morse-Bott family parametrizing Reeb orbits on M_ϵ.

One sees that item (2) is automatic since the first component of the evaluation map is surjective onto S^2. This corresponds to the fact that for any almost complex structure, an irreducible smooth almost complex conic has a (almost) complex tangent line. This assertion follows from the fact that the 2-point Gromov-Witten invariant for class H is non-zero, and that H is indecomposable. Therefore, given an almost complex conic C and a point $x \in C$, take a sequence $\{y_i\}_{i\in\mathbb{N}} \in C$ approaching x. The limit of the lines passing x and y_i is what we need. For (1) we apply Wendl's automatic transversality in dimension 4.

The virtual index of an irreducible curve $C \in \mathcal{M}^2(2[D_4]; X_0, J_\infty)$ reads:

$$ind(u) = (2-3)(2-1-1) + 0 + 0 - (0-1) = 1.$$

The computation also shows that, for generic J, the compactification of this moduli space does not contain irreducible curves with critical points or sphere bubbles since these are codimension 2 phenomena. On the other hand, we can compute $c_N(u) = 0$. Therefore, automatic transversality holds for all immersed $C \in \mathcal{M}^2_1(2[D_4]; X_0, J_\infty)$.

To show that disk bubbles do not appear, we again use Lemma 4.8 to identify the moduli space $\mathcal{M}^2_1(2[D_4]; X_0, J_\infty)$ to one on (F_4, J_1), denoted $\mathcal{M}^{crit}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$, where $\widetilde{J}$ is the extended almost complex structure. By collapsing the Reeb orbits on ∂X_0, a stable punctured disk in $\mathcal{M}^2_1(2[D_4]; X_0, J_\infty)$ descends to a stable disk in the following moduli space:

Definition 4.12. $\mathcal{M}^{crit}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$ is the moduli space of $\widetilde{J}$-holomorphic disks $u : (D^2, j) \to (\widetilde{X}_0, \widetilde{J})$ which satisfies the following:

- u has an interior marked point x and a boundary marked point y,
- $u(\partial D) \subset L$, $u(x) \in e_4$, $du(x) = 0$ with order 1.

Lemma 4.13. *Holomorphic disks* $u \in \mathcal{M}^{crit}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$ *are irreducible and transversal for generic* $\widetilde{J}$.

Proof. The argument is almost word-by-word taken from the case of open manifolds. Since u cannot develop more critical points other than x for generic choice of $\widetilde{J}$, we may apply Wendl's automatic transversality, Theorem 4.11. We have the Fredholm index:

$$\mathrm{ind}(u) = -1 + 2 \cdot 2 = 3,$$
$$c_N(u) = \frac{1}{2}(3 - 2 + 1) = 1.$$

Since we have a unique critical point of order 2,

$$3 = \mathrm{ind}(u) > c_N(u) + Z(du) = 1 + 1 = 2,$$

verifying the transversality of $u \in \mathcal{M}^{crit}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$. We now only need to show that $\partial\mathcal{M}^{\mathrm{crit}}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J}) = \emptyset$. The argument of Lemma 4.4 shows that the only possible type of reducible stable curve $u \in \partial\mathcal{M}^{\mathrm{crit}}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$ consists of a union of 2 disks in class $[D_4]$. However, given a sequence $u_k \in \mathcal{M}^{\mathrm{crit}}_1(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$ converging

to u, $u_k(x) \in e_4$ are always critical values. If a disk bubble occurs, one of the components inherits such a critical point, thus has intersection index with e_4 at least 2. But this contradicts the fact that each component is in class $[D_4]$.

□

To compute the evaluation map, we now may choose a generic path connecting $\{J^s\}_{s\in[0,1]}$ connecting $J^1 = \widetilde{J}$ and the standard toric complex structure J^0 of F_4 as in [10], while requiring e_i, $i = 1,2,3,4$ are J_t-holomorphic. In view of the arguments in Lemma 4.13, the moduli space $\mathcal{M}_1^{crit}(2[D_4]; F_4, L; J^s)$ does not develop disk or sphere bubbles. Moreover, $2[D_4]$ does not admit a multiple cover more than 2-fold, whereas these 2-fold covers are in fact curves in $\mathcal{M}_1^{crit}(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$ instead of the boundary of the moduli space $\partial\mathcal{M}_1^{\rm crit}(2[D_4]; e_4, \widetilde{X}_0, L; J^s)$. Therefore, the standard cobordism argument in [31] applies. In particular, $ev_{0*}[\mathcal{M}_1^{crit}(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})] = ev_{0*}[\mathcal{M}_1^{crit}(2[D_4]; F_4, L; J^0)]$. By Cho-Oh's classification, $\mathcal{M}_1^{crit}(2[D_4]; F_4, L; J^0)$ consists only of double covers of embedded disks in $\mathcal{M}_1([D_4]; F_4, L; J^0)$. Therefore,

$$ev_{0*}[\mathcal{M}_1^{crit}(2[D_4]; F_4, L; J^0)] = 2[L].$$

On the X_1 side, what concerns us is $\mathcal{M}^2(H'; X_1, J_\infty)$. We already saw from the argument of item (2) that these curves one-one correspond to closed curves in $\mathbb{C}P^2$ of class H which are tangent to the given embedded conic. In particular no bubbling or critical points occurs for these curves. The virtual index of such a curve C_1 is:

$$ind(C_1) = (2-3)(2-1) + 2 + 1 - 0 = 2,$$

and $c_N(C_1) = 0$. This verifies item (1). Therefore, the standard gluing arguments apply and leads to the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}^2(H'; X_1, J_\infty)\ {}_{ev^1}\times_{ev^1} \mathcal{M}_1^2(2[D_4]; X_0, J_\infty) \arrow[r, "glue"] \arrow[dr, "\widetilde{ev}_0"'] & \mathcal{M}_1(D_4'; \mathbb{C}P^2, L) \arrow[d, "ev_0"] \\
 & L
\end{tikzcd}$$

Here $\widetilde{ev}_0$ is the evaluation of $\mathcal{M}_1^2(2[D_4]; X_0, J_\infty)$ to the boundary marked points. It then follows that, for t sufficiently large,

$$deg[ev_0 : \mathcal{M}_1(D_4'; \mathbb{C}P^2, L; J_t) \to L] = 2. \tag{4.3}$$

5. Completion of the proof

In previous sections, we have computed evaluation maps of all holomorphic disks of Maslov index 2 of $(\mathbb{C}P^2, J_t)$ when t is sufficiently large. Following arguments of [20], from (4.1) we deduce that the potential function of $\mathbb{C}P^2$ in the toric degeneration picture is written as:

$$PD^u(y) = T^{u_1}y_1 + T^{u_2}y_2 + T^{4-u_1-4u_2}y_1^{-1}y_2^{-4} + 2T^{2-2u_2}y_2^{-2}.$$

By requiring $u_1 = u_2 = \frac{2}{3}$, the equation of critical points are written as:

$$\begin{cases} y_1^2 y_2^4 = 1 \\ 1 - 4y_2^{-5}y_1^{-1} - 4y_2^{-3} = 0 \end{cases} \tag{5.1}$$

We therefore deduce that $y_1 y_2^2 = \pm 1$. When we take it to be 1, we have $y_2^3 = 8$ thus clearly has three solutions. This verifies that $\mathbf{u} = (\frac{2}{3}, \frac{2}{3})$ is indeed a critical point for some values of y_1 and y_2. Moreover, it is well-known that $QH_*(\mathbb{C}P^2, \Lambda)$ is semi-simple and decomposes into 3 direct factors of Λ. Writing $QH^*(\mathbb{C}P^2; \Lambda) = \Lambda[z]/(z^3 - T)$, the idempotents are simply $\dfrac{1}{3}\epsilon_i^{-2}(z^2 + \epsilon_i z + \epsilon_i^2)$ for $i = 1,2,3$. Here ϵ_i are the roots of $x^3 - T = 0$ in Λ. Now

Corollary 2.11 implies $L(\mathbf{u})$ is indeed superheavy with respect to some partial symplectic quasi-state. This concludes our proof to Theorem 1.1.

Remark 5.1. From the above calculation, we also found 3 local systems on the exotic monotone fiber, which is different from the case of the calculation of [21], where there the monotone exotic fiber only has half number of local systems of the standard monotone fiber, i. e. the product of equators.

According to comments due to Kenji Fukaya, combining results from [1], our computation implies that this single exotic fiber is sufficient to generate certain Fukaya category with characteristic zero coefficients. However, this fiber is disjoint from $\mathbb{R}P^2$, thus cannot generate any version of Fukaya category with characteristic 2 coefficients (in fact our torus is always a zero object for characteristic 2 Fukaya categories of $\mathbb{C}P^2$). This shows that the choice of coefficient rings could be more than technical.

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